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MOTION OF A SPHERE IN AN INFINITE CONDUCTIVE FLUID, PRODUCED BY A VARIABLE MAGNETIC DIPOLE LOCATED WITHIN THE SPHERE

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§1. In [1] a study was made of two examples of turbulent flow developing in a conductive fluid under the action of an ac magnetic field. In one of these the electromagnetic field was created by a magnetic dipole $\mathbf{m}_0 e^{i\omega t}$ located in the center of a nonconductive solid sphere immersed in an infinite volume of conductive fluid. Due to the high degree of symmetry the applied electromagnetic field did not lead to directed motion of the fluid relative to the sphere.

It is of interest to consider the case of a less symmetric electromagnetic field in which the net force exerted on the fluid by the field is nonzero. Such a situation should lead to the development of a translational component in the fluid motion relative to the solid body or, what is the same, to translational motion of the solid relative to the fluid, which is at rest at infinity.

For this purpose the present study will consider the flow about a sphere, with the ac dipole displaced relative to the center of the sphere (Fig. 1).

The problem will be solved with the assumption that the eccentricity d is small in comparison to the sphere radius a,

$$\varepsilon = d/a \ll 1, \quad (1.1)$$

and that the conventional and magnetic Reynolds numbers are also small,

$$\mathrm{Re} = v_0 a/\nu \ll 1; \quad (1.2)$$

$$\mathrm{Re}_m = 4\pi\sigma v_0 a/c^2 \ll 1, \quad (1.3)$$

where v_0 is the characteristic velocity of the flow which develops and σ and ν are the conductivity and kinematic viscosity of the liquid. It should be noted that, in fact, condition (1.3) follows from Eq. (1.2), since for all conductive fluids $\nu_m = c^2/4\pi\sigma \gg \nu$.

We will consider the flow which is established after the system achieves a periodic regime.

For the case of constant electric and magnetic fields the first step in the investigation of sphere motion was made in [2], where the force of electromagnetic origin tending to set the sphere in motion relative to the conductive fluid was found (the hydrodynamic portion of the problem was not considered, although estimates were made of the effectiveness of such a means of locomotion in seawater).

§2. In view of assumption (1.3), the fluid motion exerts no effect on the electrodynamic quantities, so that the problem of defining **E** and **H** over all space is separate from the hydrodynamic problem.

The desired fields **E** and **H** are defined by a vector potential $\mathbf{A}: \mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$, $\mathbf{H} = \mathrm{rot}\,\mathbf{A}$ where in the spherical coordinate system (r, θ, α) affixed to the sphere the vector **A** has one (nonzero) component $\mathbf{A} = A(r, \theta, t)\mathbf{e}_\alpha$. Since the displacement d of the center of the

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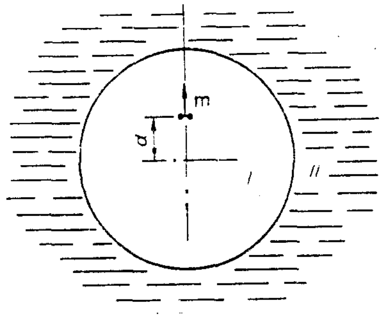

Fig. 1

dipole and the direction of the vector $\mathbf{m}_0$ coincide with the z axis, the problem is axisymmetric, i.e., $\partial/\partial\alpha \equiv 0$. The vector potential **A** satisfies the equations

$$\Delta \mathbf{A}_1 = 0; \tag{2.1}$$

$$\partial \mathbf{A}_2/\partial t = (c^2/4\pi\sigma)\Delta \mathbf{A}_2 \tag{2.2}$$

and the boundary conditions

$$A_1|_{r=a} = A_2|_{r=a}, \quad \frac{\partial A_1}{\partial r}\Big|_{r=a} = \frac{\partial A_2}{\partial r}\Big|_{r=a}, \quad A_2|_{r=\infty} \neq \infty. \tag{2.3}$$

Here the indices 1 and 2 refer to regions with the sphere I and in the conductive fluid II (see Fig. 1). Aside from condition (2.3), the solution A_1 as $r \to 0$ must have a singularity caused by the applied magnetic dipole $\mathbf{m}_0 e^{i\omega t}$.

The vector potential of a magnetic dipole displaced from the center by a distance d (Fig. 1) has the form

$$A_* = (m_0 r \sin\theta/(r^2 - 2rd + d^2)^{3/2})e^{i\omega t}.$$

With consideration of Eq. (1.1), we can rewrite this expression to an accuracy of $O(\varepsilon^2)$ in the form

$$A_* = (m_0/r^2)(\sin\theta + (3/2)\varepsilon a \sin\theta/r)e^{i\omega t}.$$

As $r \to 0$, the solution A_1 must possess such a singularity.

The periodic solution of Eqs. (2.1) and (2.2) satisfying the conditions enumerated above has the form

$$A_1 = m_0[(C_1 r + 1/r^2)\sin\theta + \varepsilon(D_1 r^2 + 3a/2r^3)\sin 2\theta]e^{i\omega t}; \tag{2.4}$$

$$A_2 = (3m_0/a^{3/2}r^{1/2})\left[C_2 H^{(2)}_{3/2}(kr)\sin\theta + \varepsilon D_2 H^{(2)}_{5/2}(kr)\sin 2\theta\right]e^{i\omega t}, \tag{2.5}$$

$$C_1 = \frac{1}{a^3}\frac{H^{(2)}_{1/2}(ka)}{H^{(2)}_{5/2}(ka)}, \quad D_1 = \frac{3}{2a^4}\frac{H^{(2)}_{3/2}(ka)}{H^{(2)}_{7/2}(ka)},$$

$$C_2 = \frac{1}{kaH^{(2)}_{5/2}(ka)}, \quad D_2 = \frac{5}{2ka}\frac{1}{H^{(2)}_{7/2}(ka)}, \quad k = (1-i)/\delta, \quad \delta = \sqrt{c^2/2\pi\sigma\omega},$$

where δ is the thickness of the skin layer; $H^{(2)}_\lambda(x)$ are Hankel functions of the second kind of order λ [3].

In Eqs. (2.4) and (2.5) the first terms correspond to a central dipole, while the terms proportional to the small parameter ε give the correction for displacement of the dipole from the center.

§3. The flow of the incompressible conductive fluid considered here is described by the hydrodynamic equations

$$\operatorname{div}\mathbf{v} = 0, \ \operatorname{rot}\mathbf{v} = \mathbf{w}; \tag{3.1}$$

$$\partial\mathbf{w}/\partial t + \nu \operatorname{rot}\operatorname{rot}\mathbf{w} = (1/\rho c)\operatorname{rot}[\boldsymbol{j} \times \mathbf{H}]. \tag{3.2}$$

In the latter equation the nonlinear term rot $[\mathbf{v} \times \mathbf{w}]$ is omitted, since it is small in comparison with the viscous term as shown by Eq. (1.2).

The Lorentz force rotor $\mathbf{f} = (1/c)[\mathbf{j} \times \mathbf{H}] = (\sigma/c)[\mathbf{E}_2 \times \mathbf{H}_2]$ is calculated from solution (2.5) for the vector potential in the region occupied by the conductive fluid. To simplify the formulas, we will consider below the case of strong skin effect, i.e.,

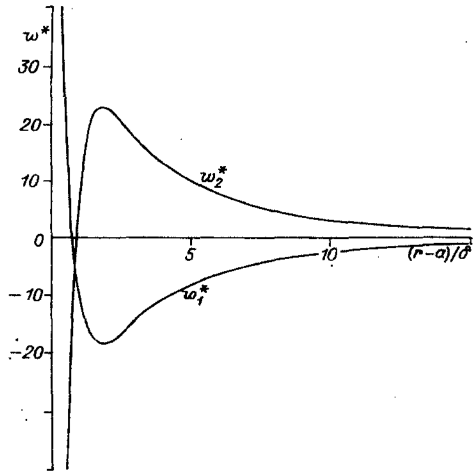

Fig. 2

$$\delta/a \ll 1. \tag{3.3}$$

In this case the Lorentz forces are concentrated in a thin layer of fluid adjoining the sphere surface, while

$$\text{rot}\,\mathbf{f} = -(1/2\pi)(m_0/a^3)^2(1/a\delta)(a/r)^3 e^{-2(r-a)/\delta}\ [(9/4)\sin 2\theta + (45/2)\varepsilon \sin\theta\cdot(3\cos^2\theta - 1)\mathbf{e}_\alpha] + \text{an oscillating supplement.} \tag{3.4}$$

A term proportional to the square of the small parameter ε has been omitted in Eq. (3.4).

The effect of the forces on the fluid, as in the case of a central dipole, consists of stationary and oscillating components. Because of the small velocity of the oscillating flow [1], the stationary flow is of major interest, so that the oscillating component of the force field was not written out in Eq. (3.4).

The first term in Eq. (3.4), dependent on angle θ as $\sin 2\theta$, corresponds to the force field from a central dipole as studied in [1]. The second term, proportional to $\sin\theta\cdot(3\cos^2\theta - 1)$, is a small correction caused by the parameter ε. We will now denote this component of the force field by $\mathbf{f}_1$:

$$\text{rot}\,\mathbf{f}_1 = -(45/4\pi)\varepsilon(m_0/a^3)^2(1/a\delta)(a/r)^3 e^{-2(r-a)/\delta}\sin\theta\cdot(3\cos^2\theta - 1)\mathbf{e}_\alpha. \tag{3.5}$$

The flow produced by the forces $\mathbf{f}_1$ will be considered in the present study. It should be noted that in the approximation of Eq. (3.3), the force $\mathbf{f}_1$ has only a single component,

$$\mathbf{f}_1 = \frac{45}{4\pi}\varepsilon\frac{H_0^2}{\delta}\left(\frac{a}{r}\right)^2 e^{-2(r-a)/\delta}\sin^2\theta\cdot\cos\theta\cdot\mathbf{e}_r; \quad H_0 = m_0/a^3 \tag{3.6}$$

(H_0 is the characteristic magnetic field intensity).

The flow under study has axial symmetry, while $v_\alpha \equiv 0$. The flow turbulence vector has only an α-component, i.e., $\mathbf{w} = w(r,\theta)\mathbf{e}_\alpha$. According to Eqs. (3.2) and (3.5), the equation for $w(r, \theta)$ has the form

$$\frac{\partial^2(rw)}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot w)\right] = \Phi(r)(2\sin\theta - 3\sin^3\theta), \tag{3.7}$$

$$\Phi(r) = \frac{45}{4\pi}\varepsilon\frac{H_0^2}{\rho a\delta}\left(\frac{a}{r}\right)^3 e^{-2(r-a)/\delta}.$$

The solution of Eq. (3.7) finite at infinity takes on the form

$$w(r,\theta) = w_1(r)\sin\theta + w_2(r)\sin 2\theta,$$
$$w_1(r) = \beta_1 r^{-2} - 0{,}8\beta_2 r^{-4} - (2/15)[rJ_0(r) - r^{-2}J_3(r)] + (12/35)[r^3J_{-2}(r) - r^{-4}J_5(r)],$$
$$w_2(r) = \beta_2 r^{-4} - (3/7)[r^3J_{-2}(r) - r^{-4}J_5(r)]; \tag{3.8}$$

$$J_\mu(r) = \int_\infty^r \xi^\mu\,\Phi(\xi)\,d\xi. \tag{3.9}$$

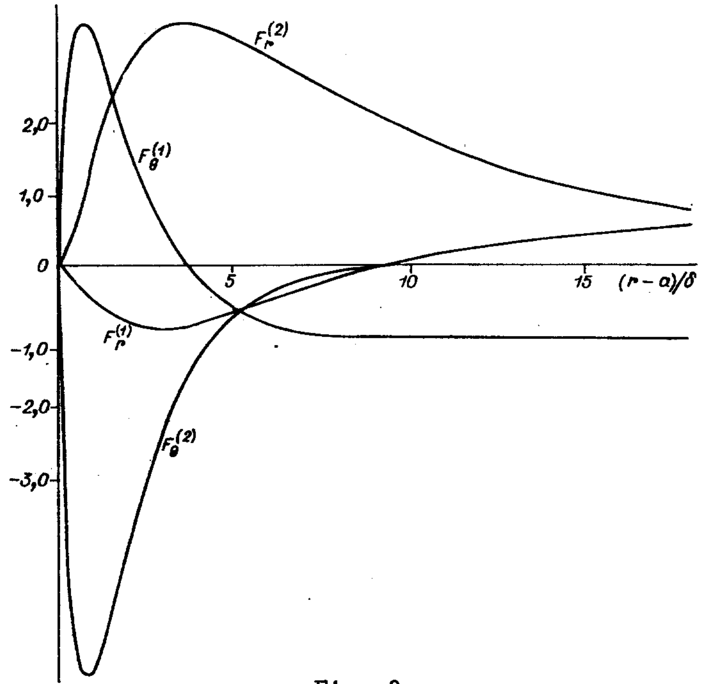

Fig. 3

The arbitrary constants β_1 and β_2 will be defined below.

§4. The velocity field is determined from Eqs. (3.1), the first of which is identically satisfied by the introduction of a vector potential ψ for the velocity $\mathbf{v} = \mathrm{rot}\,[\psi(r, \theta)\mathbf{e}_\alpha]$:

$$v_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot\psi), \quad v_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r\psi). \tag{4.1}$$

From the second of Eqs. (3.1) and the solution (3.8) we have

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\cdot\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin\theta)\right] = -[w_1(r)\sin\theta + w_2(r)\sin^3\theta]. \tag{4.2}$$

The boundary conditions for $\psi(r, \theta)$ stemming from the conditions for the velocity vector $\mathbf{v}$ have the form

$$\psi(r, \theta)|_{r=a} = 0, \quad \left.\frac{\partial\psi(r, \theta)}{\partial r}\right|_{r=a} = 0; \tag{4.3}$$

$$\frac{1}{r}\psi(r, \theta)|_{r\to\infty} \neq \infty. \tag{4.4}$$

The latter condition corresponds to finiteness of the flow velocity as $r \to \infty$.

The solution of Eq. (4.2) takes on the form

$$\psi(r, \theta) = P(r)\sin\theta + Q(r)\sin^3\theta; \tag{4.5}$$

$$\begin{aligned} P(r) &= \alpha_1 r + \alpha_2 r^{-2} - 0.8\alpha_3 r^{-4} + (1/2)\beta_1 + (1/15)[J_3(r) - rJ_2(r)] + \\ &+ (2/105)[r^{-4}J_7(r) - r^5 J_{-2}(r)] + (1/21)[r^3 J_0(r) - r^{-2}J_5(r)]; \\ Q(r) &= \alpha_3 r^{-4} + 0.1\beta_2 r^{-2} + (1/42)[r^5 J_{-2}(r) - r^{-4}J_7(r)] + (3/70)[r^{-2}J_5(r) - r^3 J_0(r)]. \end{aligned} \tag{4.6}$$

Here we present a solution satisfying Eq. (4.4). From condition (4.3) it follows that

$$P(a) = 0,\ Q(a) = 0,\ P'(a) = 0,\ Q'(a) = 0. \tag{4.7}$$

From this it is evident that the kinematic conditions (4.3) are insufficient for definition of the arbitrary constants α_1, α_2, α_3, β_1, and β_2.

The insufficiency of the conditions stems from the stationary nature of the flow, as a result of which the force Σ_z acting on the dipole $\mathbf{m}_0 e^{i\omega t}$ due to the magnetic field of currents in the fluid compensates the equalizing stresses T_z acting on the sphere surface and produced by the fluid, i.e.,

$$\Sigma_z + T_z = 0. \tag{4.8}$$

The force Σ_z is equal in magnitude to the compensating electromagnetic volume forces within the entire fluid, while it is opposite in direction; consequently,

$$\Sigma_z = -2\pi \int_0^{\pi}\int_a^{\infty} f_{1r}(r,\ \theta)\, r^2 \cos\theta \cdot \sin\theta \cdot d\theta dr.$$

Substituting Eq. (3.6) here, we obtain

$$\Sigma_z = -3\varepsilon a^2 H_0^2. \tag{4.9}$$

The expression for T_z is found from consideration of the stresses acting on the sphere surface. The result has the form

$$T_z = 4\pi(-\beta_1 \rho\nu + (3/4\pi)\,\varepsilon a^2 H_0^2). \tag{4.10}$$

From Eqs. (4.8)-(4.10) it follows that $\beta_1 = 0$.

The remaining constants are found from conditions (4.7) and take on the following values:

$$\alpha_1 = -(2/45)a^{-1}J_3(a) + (1/15)J_2(a) - (1/45)a^2 J_0(a),$$
$$\alpha_2 = (3/35)a^7 J_{-2}(a) - (1/9)a^5 J_0(a) - (1/45)a^2 J_3(a) + (1/21)J_5(a),$$
$$\alpha_3 = (1/12)a^9 J_{-2}(a) + (1/42)J_7(a) - (3/28)a^7 J_0(a),$$

$$\beta_2 = -(15/14)a^7 J_{-2}(a) - (3/7)J_5(a) + (3/2)a^5 J_0(a).$$

The values of $J_\mu(a)$ appearing here are defined by Eq. (3.9).

It must be noted that the constant α_1 appearing in Eqs. (4.6) determines the velocity of the liquid at infinity. In fact, according to Eqs. (4.1) and (4.5),

$$\mathbf{v}|_{r=\infty} = 2\alpha_1(\cos\theta \cdot \mathbf{e}_r - \sin\theta \cdot \mathbf{e}_\theta) = 2\alpha_1 \mathbf{e}_z.$$

Thus, far from the sphere surface the fluid has a translational velocity u_0 directed along the z axis (in the coordinate system fixed to the sphere), while

$$u_0 = 2\alpha_1.$$

§5. We will use the formulas obtained above to investigate the flow. We note that the integrals $J_\mu(r)$ [Eq. (3.9)] appearing in the expressions for $w(r,\ \theta)$ and $\psi(r,\ \theta)$ can be expanded in an asymptotic series in powers of the small parameters δ/a, when we consider the exponential decay of the integrand $\Phi(r)$ (the integrals J_3, J_5, and J_7 are calculated exactly). From this we can obtain an expression for the velocity u_0:

$$u_0 = 2\alpha_1 \cong \varepsilon K(\delta^2/a^2)(1 - 3.5\delta/a + \ldots),\quad K = 3aH_0^2/8\pi\rho\nu. \tag{5.1}$$

For example, for $\delta/a = 0.1$ the terms written in Eq. (5.1) give $u_0 = 0.00654\ \varepsilon K$. Numerical integration gives a value of 0.00783, i.e., the error in the approximation of Eq. (5.1) comprises about 10%.

It is natural that in Eq. (1.1) the velocity of the flow incident upon the sphere is directly proportional to the amount of displacement of the dipole from the center of the sphere. Since the sign in Eq. (5.1) is positive, the direction of this flow coincides with the z axis (Fig. 1). Consequently, the sphere moves in a reverse direction relative to the fluid at rest at infinity. For the case being considered $\delta \ll a$ the velocity u_0 is proportional to δ^2 (just as is the flow velocity in the case of a central dipole [1]). In particular, as $\delta \to 0$, $u_0 \to 0$. At first glance it seems that this result contradicts Eq. (4.9), according to which the "attractive force" produced on the sphere by the electromagnetic field is independent of δ. In fact, however, at $\delta = 0$ the force Σ_z just compensates the surface pressure forces produced by the electromagnetic field; volume nonpotential forces are absent in the fluid, so that the force Σ_z compensates the fluid pressure gradient and does not lead to fluid motion.

Figures 2 and 3 depict the character of the flow under study for $\delta/a = 0.1$. Figure 2 shows the dimensionless functions $w_1^* = w_1(r)a/u_0$ and $w_2^* = w_2(r)a/u_0$, which, according to Eqs. (3.8), define the turbulence distribution. Figure 3 shows the dimensionless functions $F_r^{(1)}$, $F_r^{(2)}$, $F_\theta^{(1)}$, and $F_\theta^{(2)}$ which defines the velocity fields according to the formulas

$$v_r(r,\ \theta) = u_0(F_r^{(1)} \cos\theta + F_r^{(2)} \sin^2\theta \cdot \cos\theta),$$
$$v_\theta(r,\ \theta) = u_0(F_\theta^{(1)} \sin\theta + F_\theta^{(2)} \sin^3\theta),$$

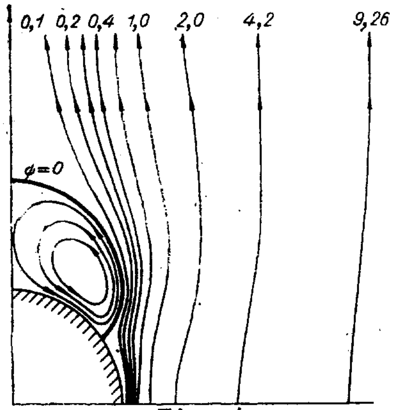

Fig. 4

obtained from Eqs. (4.1) and (4.5). It is evident from Figs. 2 and 3 that over the entire flow field the flow velocity is of the order of magnitude of the incident flow. Consequently, for the characteristic velocity v_0 in Eq. (1.2) we may use u_0. Now, on the basis of Eq. (5.1), condition (1.2) transforms to the inequality

$$\frac{1}{4\pi}\,\varepsilon\left(\frac{\delta}{a}\right)^2\frac{a^2H_0^2}{\rho\nu^2} \ll 1,$$

which limits the value of the applied magnetic field.

Figure 4 shows flow lines [lines of constant value of $r\sin\theta\cdot\psi(r,\theta)$] lying in the plane α = const. The flow lines are shown only for the upper hemisphere, since the flow in the lower hemisphere is completely symmetric with respect to the plane $z = 0$. In Fig. 4 one can clearly see the intensity of the turbulence localized in some finite region adjacent to the sphere surface. The intensity of the turbulence may be judged from the behavior of w_1^*, w_2^* in Fig. 2. The general picture of the flow is that of an external flow around some effective body (indicated by the heavy line).

In conclusion, we note that the force Σ_z setting the sphere in motion with a translational velocity u_0 is approximately $(a/\delta)^2$ times larger than the Stokes force $\mathscr{F} = 6\pi\rho\nu a u_0$.

In fact, according to Eqs. (4.9) and (5.1),

$$\mathscr{F}/\Sigma_z \approx (3/4)\delta^2/a^2.$$

The difference between these forces is caused by the difference in the flow patterns for the case under consideration and that of Stokes flow-by.

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